XMMS Equalizer Formula, Simplified

Assuming a passband amplitude G_1, a cutoff amplitude G_2, a center frequency f_0, a sampling frequency F_s, and a bandwidth N in octaves, an equalizer node with digital frequency $\theta = \frac{2\pi f_0}{F_s}$ is denoted by x_0 where

$$\beta_2 \left(G_1, G_2, N, \theta\right) {x_0}^2 + \beta_1 \left(G_1, G_2, N, \theta\right) x_0 + \beta_0 \left(G_1, G_2, N, \theta\right) = 0,$$

where

- $\beta_2 \left(G_1, G_2, N, \theta\right) = {G_2}^2 \cos^2 \theta - 2{G_2}^2 \cos\theta \cos\left(2^{-N/2}\theta\right) + {G_2}^2 - {G_1}^2 \sin^2\left(2^{-N/2}\theta\right),$
- $\beta_1 \left(G_1, G_2, N, \theta\right) = 2{G_2}^2 \cos^2\left(2^{-N/2}\theta\right) + {G_2}^2 \cos^2\theta - 2{G_2}^2 \cos\theta\cos\left(2^{-N/2}\theta\right) - {G_2}^2 + {G_1}^2 \sin^2\left(2^{-N/2}\theta\right)$, and
- $\beta_0 \left(G_1, G_2, N, \theta\right) = \frac{1}{4}{G_2}^2 \cos^2\theta - \frac{1}{2}{G_2}^2 \cos\theta\cos\left(2^{-N/2}\theta\right) + \frac{1}{4}{G_2}^2 - \frac{1}{4}{G_1}^2 \sin^2\left(2^{-N/2}\theta\right).$

For simplicity, we will always have a passband amplitude of 0 dB and a cutoff amplitude of -3 dB. In other words, $G_1 = 1$ and $G_2 = \frac{1}{\sqrt{2}}$. Thus, the simpler formula for x_0 can be denoted by

$$\beta_2 \left(N, \theta\right) {x_0}^2 + \beta_1 \left(N, \theta\right) x_0 + \beta_0 \left(N, \theta\right) = 0,$$

where

- $\beta_2 \left(N, \theta\right) = \frac{1}{2}\cos^2\theta - \cos\theta\cos\left(2^{-N/2}\theta\right) + \frac{1}{2} - \sin^2\left(2^{-N/2}\theta\right),$
- $\beta_1 \left(N, \theta\right) = \cos^2\left(2^{-N/2}\theta\right) + \frac{1}{2}\cos^2\theta - \cos\theta\cos\left(2^{-N/2}\theta\right) - \frac{1}{2} + \sin^2\left(2^{-N/2}\theta\right)$
  $= \frac{1}{2}\cos^2\theta - \cos\theta\cos\left(2^{-N/2}\theta\right) + \frac{1}{2}$, and
- $\beta_0 \left(N, \theta\right) = \frac{1}{8}\cos^2\theta - \frac{1}{4}\cos\theta\cos\left(2^{-N/2}\theta\right) + \frac{1}{8} - \frac{1}{4}\sin^2\left(2^{-N/2}\theta\right).$

These formulas can be simplified to produce

- $\beta_2 \left(N, \theta\right) = \frac{1}{2}\left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)^2 - \frac{1}{2}\sin^2\left(2^{-N/2}\theta\right),$
- $\beta_1 \left(N, \theta\right) = \frac{1}{2}\left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)^2 + \frac{1}{2}\sin^2\left(2^{-N/2}\theta\right)$, and
- $\beta_0 \left(N, \theta\right) = \frac{1}{8}\left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)^2 - \frac{1}{8}\sin^2\left(2^{-N/2}\theta\right).$

x_0 can be found using the quadratic formula:

$$x_0 = \frac{-\beta_1\left(N, \theta\right) \pm \sqrt{\left[\beta_1\left(N, \theta\right)\right]^2 - 4\beta_0\left(N, \theta\right)\beta_2\left(N, \theta\right)}}{2\beta_2\left(N, \theta\right)}.$$

Note that $\beta_1\left(N, \theta\right) = \beta_2\left(N, \theta\right) + \sin^2\left(2^{-N/2}\theta\right)$. Thus,

$$x_0 = -\frac{1}{2} + \frac{-\sin^2\left(2^{-N/2}\theta\right) \pm \sqrt{\left[\beta_1\left(N, \theta\right)\right]^2 - 4\beta_0\left(N, \theta\right)\beta_2\left(N, \theta\right)}}{2\beta_2\left(N, \theta\right)}.$$

Examining the discriminant,

$$\left[\beta_1\left(N, \theta\right)\right]^2 - 4\beta_0\left(N, \theta\right)\beta_2\left(N, \theta\right) = \frac{1}{4}\left[\left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)^2 + \sin^2\left(2^{-N/2}\theta\right)\right]^2 - \frac{1}{4}\left[\left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)^2 - \sin^2\left(2^{-N/2}\theta\right)\right]^2.$$

This simplifies to

$$[\beta_1(N,\theta)]^2 - 4\beta_0(N,\theta)\,\beta_2(N,\theta) = \left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)^2 \sin^2\left(2^{-N/2}\theta\right).$$

Note that θ is valid only if it is non-negative and does not exceed the Nyquist frequency. That is, $0 \leq \theta \leq \pi$. This also means that $\cos\left(2^{-N/2}\theta\right) \geq \cos\theta$ and $\sin\left(2^{-N/2}\theta\right) \geq 0$. Hence,

$$\sqrt{[\beta_1(N,\theta)]^2 - 4\beta_0(N,\theta)\,\beta_2(N,\theta)} = \left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)\sin\left(2^{-N/2}\theta\right).$$

The quadratic formula now consists of

$$x_0 = -\frac{1}{2} + \frac{\sin\left(2^{-N/2}\theta\right)\left[-\sin\left(2^{-N/2}\theta\right) \pm \left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)\right]}{2\beta_2(N,\theta)}.$$

At this point, we can now plug in the value of $\beta_2(N,\theta)$:

$$x_0 = -\frac{1}{2} + \frac{\sin\left(2^{-N/2}\theta\right)\left[-\sin\left(2^{-N/2}\theta\right) \pm \left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)\right]}{\left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)^2 - \sin^2\left(2^{-N/2}\theta\right)}.$$

To make the quadratic formula look more elegant and easier to analyze, we can negate both the numerator and the denominator of the second term:

$$x_0 = -\frac{1}{2} + \frac{\sin\left(2^{-N/2}\theta\right)\left[\sin\left(2^{-N/2}\theta\right) \mp \left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)\right]}{\sin^2\left(2^{-N/2}\theta\right) - \left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)^2}.$$

Because the denominator is a difference of squares, we can simplify the quadratic formula into:

$$x_0 = -\frac{1}{2} + \frac{\sin\left(2^{-N/2}\theta\right)}{\sin\left(2^{-N/2}\theta\right) \pm \left(\cos\left(2^{-N/2}\theta\right) - \cos\theta\right)}.$$

The correct value of x_0 is the smaller value expressed by the quadratic formula. Taking into account the fact that $\cos\left(2^{-N/2}\theta\right) \geq \cos\theta$ and $\sin\left(2^{-N/2}\theta\right) \geq 0$, we must choose the plus sign:

$$x_0 = -\frac{1}{2} + \frac{\sin\left(2^{-N/2}\theta\right)}{\sin\left(2^{-N/2}\theta\right) + \cos\left(2^{-N/2}\theta\right) - \cos\theta}.$$